THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1520G/H University Mathematics 2014-2015
Suggested Solution to Assignment 1

Exercise 10.1

(31) $\lim_{n\to\infty} \dfrac{1-5n^4}{n^4+8n^3} = \lim_{n\to\infty} \dfrac{\frac{1}{n^4}-5}{1+\frac{8}{n}} = -5.$

(45) Note that $-\dfrac{1}{n} \leq \dfrac{\sin n}{n} \leq \dfrac{1}{n}$ for all natural numbers n, and $\lim_{n\to\infty} -\dfrac{1}{n} = \lim_{n\to\infty} \dfrac{1}{n} = 0.$
By sandwich theorem, $\lim_{n\to\infty} \dfrac{\sin n}{n} = 0.$

(69) $$\lim_{n\to\infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{(\frac{3n-1}{2})}\right)^n = \lim_{n\to\infty} \left[\left(1+\frac{1}{(\frac{3n-1}{2})}\right)^{\frac{3n-1}{2}}\right]^{2/3} \left(1+\frac{1}{(\frac{3n-1}{2})}\right)^{1/3} = e^{2/3}$$

Exercise 2.2

(33) $\lim_{u\to 1} \dfrac{u^4-1}{u^3-1} = \lim_{u\to 1} \dfrac{(u-1)(u+1)(u^2+1)}{(u-1)(u^2+u+1)} = \lim_{u\to 1} \dfrac{(u+1)(u^2+1)}{u^2+u+1} = \dfrac{4}{3}.$

(39) $\lim_{x\to 2} \dfrac{\sqrt{x^2+12}-4}{x-2} = \lim_{x\to 2} \dfrac{\sqrt{x^2+12}-4}{x-2} \cdot \dfrac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4} = \lim_{x\to 2} \dfrac{x+2}{\sqrt{x^2+12}+4} = \dfrac{1}{2}.$

1. Note, by the construction of the sequence, we have $a_n \geq 0$ for all natural numbers n.

   Using mathematical induction to show that $a_n \leq 3$ for all natural numbers n.
   (1) When $n=1$, $a_1 = 1 \leq 3$.
   (2) Assume $a_k \leq 3$ for some natural number k. Then, we have

$$\begin{aligned} a_k &\leq 3 \\ \frac{1}{16} &\leq \frac{1}{a_k+13} \\ -9 &\geq -\frac{144}{a_k+13} \\ 3 &\geq 12 - \frac{144}{a_k+13} \\ 3 &\geq a_{k+1} \end{aligned}$$

   By mathematical induction, $a_n \leq 3$ for all natural numbers n.

(b) We have, for $n > 1$

$$\begin{aligned} a_n - a_{n-1} &= \frac{12a_{n-1}+12}{a_{n-1}+13} - a_{n-1} \\ &= \frac{-a_{n-1}^2 - a_n + 12}{a_{n-1}+13} \\ &\geq 0 \end{aligned}$$

Note that $a_{n-1} \leq 3$, so $-a_{n-1}^2 - a_n + 12 \geq 0$.

By monotonic sequence theorem, $\{a_n\}$ converges. Suppose $\lim_{n\to\infty} a_n = A$, we have

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{12a_{n-1}+12}{a_{n-1}+13} \\
A &= \frac{12A+12}{A+13} \\
A &= 3
\end{aligned}$$

Note, $A = -4$ is rejected.

2. Note that $\frac{1}{\sqrt{n^2+n}} \leq \frac{1}{\sqrt{n^2+i}} \leq \frac{1}{\sqrt{n^2}} = \frac{1}{n}$ for all $1 \leq i \leq n$, so we have

$$\frac{1}{\sqrt{n^2+n}} \cdot n \leq \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \leq \frac{1}{n} \cdot n = 1$$

Note that $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$. By sandwich theorem,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} = 1.$$